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Examples and counterexamples for Riemann sums

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ABSTRACT

The object of this paper is to present some new results on almost everywhere convergence of Riemann sums. We give an alternative proof of a theorem of L. Dubins and J. Pitman, extending an earlier result of B. Jessen. Then, we show that the latter theorem is nearly optimal. Finally, we treat some related questions.

1. INTRODUCTION

Let f be a measurable function on $\mathbb{T} = [0,1[= \mathbb{R}/\mathbb{Z}$. For $n = 1, 2, \cdots$ define as follows the Riemann sums operators

$$(1)\qquad \forall x \in \mathbb{T}, \quad R_n(f)(x) = \frac{1}{n} \sum_{0 \le j < n} f\left(x + \frac{j}{n}\right).$$

Let m denote the Lebesgue measure on $\mathbb{T}$. For $f \in L^1(m)$, it is a well-known fact that $\{R_n(f), n \ge 1\}$ converges to $\int_{\mathbb{T}} f\, dm$ in the mean. But as was shown by W. Rudin [12], almost sure convergence may fail, even for bounded functions. According to ([12], Theorem).

Theorem A. *Suppose S is an increasing sequence of positive integers satisfying the following*

$$(2)\qquad \begin{cases} \text{for any } N \ge 1, \text{ there is a set } S_N \text{ of } N \text{ elements of } S, \text{ none of which} \\ \quad \text{divides the least common multiple (l.c.m.) of the others.} \end{cases}$$

Then there is a measurable subset A of $\mathbb{T}$, such that if $f = \mathbf{1}_A$, $\{R_n(f), n \in \mathcal{S}\}$ does not converge almost everywhere.

Consequently, there is no maximal inequality for such Riemann sums. Indeed, otherwise this would imply by means of Banach Principle ([7], Chapter 1) that the set of elements of $L^2(m)$ for which $\{R_n(f), n \geq 1\}$ converges almost everywhere is closed. But, $\{R_n(f), n \geq 1\}$ does converge almost everywhere when f is a finite linear combination of the characters $e_n(x) = \exp(2i\pi nx)$, $n \in \mathbb{Z}$, so that this set is also everywhere dense in $L^2(m)$, therefore contradicting the theorem of W. Rudin.

Theorem A also complements the theorem of B. Jessen [6] asserting that $\{R_{n_k}(f), n \geq 1\}$ is almost surely convergent for $f \in L^1(m)$ whenever $\mathcal{S} = \{n_k, k \geq 1\}$ is an increasing sequence of positive integers satisfying

$$\forall k \geq 1, \quad n_k \text{ divides } n_{k+1}. \tag{3}$$

Following L. Dubins and J. Pitman [4], we call such a sequence a *chain*. More generally, for sets of positive integers $\mathcal{S}_1, \cdots, \mathcal{S}_d$, set

$$[\mathcal{S}_1, \cdots, \mathcal{S}_d] = \{[n_1, \cdots, n_d] \,|\, n_i \in \mathcal{S}_i, \ i = 1, \cdots, d\}, \tag{4}$$

where $[n_1, \cdots, n_d]$ denotes the *l.c.m.* of $n_1, \cdots, n_d$.

For a set of positive integers $\mathcal{S}$, by its *dimension* we mean the least positive integer d such that $\mathcal{S}$ is a subset of $[\mathcal{S}_1, \cdots, \mathcal{S}_d]$ for chains $\mathcal{S}_1, \cdots, \mathcal{S}_d$. The theorem of B. Jessen was extended by L. Dubins and J. Pitman [4]. They proved the following theorem.

Theorem B. *If $\mathcal{S}$ has dimension d and $f \in L(\log^+ L)^{d-1}$, then*

$$m\left\{x \,\Big|\, \lim_{\mathcal{S} \ni n \to \infty} R_n(f)(x) = \int_{\mathbb{T}} f \, dm\right\} = 1. \tag{5}$$

Moreover, there exists a constant C_d depending on d only such that

$$\forall B > 0, \quad m\{x \in \mathbb{T} \,|\, M_{\mathcal{S}}(f)(x) > B\} \leq \frac{C_d}{B} \int_{\mathbb{T}} |f|(\log^+ |f|)^{d-1} \, dm. \tag{6}$$

Recall that $L(\log^+ L)^{d-1}$ denotes the set of Lebesgue measurable functions on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} |f|(\log^+ |f|)^{d-1} \, dm < \infty,$$

being understood that $\log^+ x = \log_e x$ if $x \geq e$ and equals 1 for $0 \leq x \leq e$. A partial result ($d = 2, f$ bounded) was proved by R.C. Baker [1].

Recently, R. Nair [10] suggested a more elementary proof avoiding the use of martingale theory. His argument is based on dominated estimates ([7], 1.6, p. 50), Baker's observation that

$$R_{[m,n]}(f) = R_m(R_n(f)), \tag{7}$$

and an induction argument on the dimension of $\mathcal{S}$, which is incorrect. But, it is easy to arrange the proof so that Nair's idea is tractable. The referee, however suggested an elegant alternative proof which is displayed in Section 2.

Afterwards, we prove that for no integer $d \geq 2$ and for no $0 < \varepsilon < 1$ can $L(\log^+ L)^{d-1}$ in Theorem B be replaced by $L(\log^+ L)^{d-1-\varepsilon}$, which answers a question of L. Dubins and J. Pitman [4]. This assertion for $d = 2$ and $\varepsilon = 1$ is due to R.C. Baker and we show how one can modify his arguments, based on an elementary but rather technical lemma, in order to obtain the desired result.

Section 4 is concerned with a new proof, mainly due to J. Bourgain [2], of Theorem A and with the study of the convergence of Riemann sums along sets of integers, which neither fulfill (2), nor are of finite dimension. We extend a construction of L. Dubins and J. Pitman to exhibit some such sets and provide counterexamples, where we do not have almost sure convergence. The main tool is a lemma of R.C. Baker [1], which gives a necessary condition on the growth of the sequence to have convergence. After these negative results, we state, following J. Marcinkiewicz and R. Salem [8], a condition on the coefficients of the Fourier development of $f \in L^2(m)$ which ensures the convergence.

In all the rest of this paper, we adopt the following convention of language.

Definition A. Let $\mathcal{L}$ be a set of functions and let (n_k) be a set of integers. We say that (n_k) is a $\mathcal{L}$-sequence if for every function f in $\mathcal{L}$ the sequence $\{R_{n_k}(f) \mid k \geq 1\}$ converges almost everywhere.

Acknowledgement. We would like to thank the referee for his numerous interesting remarks. Among other things, the following proof of Theorem B is due to him.

2. AN ALTERNATIVE PROOF OF THEOREM B

Using the property that $R_n(f)$ is $(1/n)$-periodic and thus $(1/m)$-periodic for $m \mid n$, B. Jessen proved ([6], Lemma 2) a maximal inequality for Riemann sums indexed on chains, which can be stated as follows.

Lemma 1. *Suppose $f \in L^1(m)$ and let $\mathcal{S}$ be a chain. Let*

$$M_{\mathcal{S}}(f) = \sup_{s \in \mathcal{S}} |R_s(f)|. \tag{8}$$

Then,

$$\forall B > 0, \quad m\{x \in \mathbb{T} \mid M_{\mathcal{S}}(f)(x) > B\} \leq \frac{1}{B} \int_{\mathbb{T}} f\, dm. \tag{9}$$

If f is a continuous function then $\lim_{n \to \infty} R_n f(x) = \int_{\mathbb{T}} f$ for every x. Since the set of continuous functions is dense in $L \log^{d-1} L$, by the Banach Principle, we just need to prove an appropriate maximal inequality.

As a consequence of $R_{[p,q]} f = R_p \circ R_q f$, we have

$$(10)\qquad \sup_k |R_{n_k} f| \le T_1 \circ \ldots \circ T_d f,$$

where $T_i f = \sup_{n \in S_i} |R_n f|$. Hence we just need to prove the maximal inequality

$$(11)\qquad m\{T_1 \circ \ldots \circ T_d f > \lambda\} \le \frac{C}{\lambda}\left(\int |f|(\log^+ |f|)^{d-1} + 1\right),$$

with C independent of f.

By Lemma 1, we need to show that $T_2 \circ \ldots \circ T_d$ is a bounded $L \log^{d-1} L \to L^1$ map.

By Theorem 4.34 in Chapter XII, page 118, of [13] we have, for every $i = 1, \ldots, d$ and $r \ge 0$,

$$(12)\qquad \int T_i g(\log^+ T_i g)^r \le C_r \int g(\log^+ g)^{r+1} + C_r,$$

with C_r depending on r only. It follows by induction on d from (12) that

$$(13)\qquad \int T_2 \circ \ldots \circ T_d f \le C \int f(\log^+ f)^{d-1} + C.$$

3. OPTIMALITY OF THEOREM B

In this section, we show that Theorem B is nearly best possible, more precisely, for all integers $d \ge 2$ and for all ε with $0 < \varepsilon < 1$, we exhibit a set K of dimension d and a function f in $L(\log^+ L)^{d-1-\varepsilon}$ such that $\{R_n(f), n \in K\}$ does not converge almost everywhere. Thus, we answer a question posed by L. Dubins and J. Pitman [4]. Our proof is based on the work of R.C. Baker [1], who showed that the sequence $(2^k 3^l)$, which is of dimension 2, is not an L-sequence, and hence $L(\log^+ L)$ in Theorem B (case $d = 2$) cannot be replaced by L. However, it seems to us that his proof is incomplete. Fortunately, Baker's main lemma remains valid under strengthened hypothesis and we are able to generalize his result in order to prove the following statement.

Theorem 1. *For any integer $d \ge 2$ and for any real number ε with $0 < \varepsilon < 1$, there exist a sequence (n_k) of dimension d and a function $f \in L(\log^+ L)^{d-1-\varepsilon}$, such that $R_{n_k}(f)$ does not converge almost everywhere. Hence, Theorem B is nearly sharp.*

Remark. Theorem 1 above does not answer precisely whether Theorem B is optimal or not. Indeed, we would like to prove that, for sequences of dimension d, the space $L(\log^+ L)^{d-1}$ is the largest Orlicz space for which almost everywhere convergence holds. This would mean that if $\omega(x)$ is a positive increasing function with

$$(14)\qquad \lim_{x \to \infty} \frac{\omega(x)}{\log^{d-1} x} = 0,$$

then we can find some sequence (n_k) and some $f \in L_\omega$ such that $R_{n_k}(f)$ does not converge almost everywhere. Unfortunately, our method of proof does not lead to such a definitive result.

Our proof proceeds in two steps. First, we need a corrected version of Lemma 3.1 of [1].

Lemma 3. *Let $b_1 \leq b_2 \leq \ldots$ be a non-decreasing sequence of positive integers, such that there exist $0 < \eta < 1$ and $c_1 > 0$ satisfying*

$$c_1^{-1} r^{1-\eta} < b_r < c_1 r^{1-\eta}, \quad r \geq 1.$$

Then there is a convergent series $\sum_{r=1}^{\infty} \psi_1(r)$ of decreasing positive terms and a divergent series $\sum_{r=1}^{\infty} \psi(r)$ of decreasing positive terms, such that

$$(15) \qquad \begin{cases} (4rb_r^2)^{-1} \leq \psi(r) \leq \dfrac{1}{2} \\ \text{and } \psi(r) = o(\psi_1(b_r)). \end{cases}$$

Moreover, if we extend ψ_1 to $[1, \infty[$ in such a way that ψ_1 is affine on the intervals $[r, r+1]$, then, letting $f(x) := x\psi_1(\log^{\delta} x)$, with $\delta \geq 1$ a real number, there exist c_2 and c_3, depending only on δ, such that $f(y) > c_2 f(x)$, whenever $c_3 < x < y$.

Proof. We follow Baker and define inductively two sequences of integers $N_1 < N_2 < \ldots$ and $M_1 < M_2 < \ldots$. Note that our hypothesis about the sequence $(b_r)_r$ implies that there exists $c_4 > 0$ such that

$$(16) \qquad c_4^{-1} r^{1/(1-\eta)} < l < c_4 r^{1/(1-\eta)}$$

whenever $b_l = r$.

Let $\rho > 2$ be such that $c_4^{-1} x^{1/(1-\eta)} - c_4 > x - 1$ for all real numbers $x \geq \rho$. Choose $N_1 = [2^{1/\eta}] + 1$ and $M_1 = 1$, and suppose that $N_1, \ldots, N_k$ and $M_1, \ldots, M_k$ have been defined in such a way that, for $2 \leq j \leq k$, the following conditions hold:

(i) $b_{M_j - 1} < N_j = b_{M_j}$
(ii) $M_j - M_{j-1} > 2^{j-1}(N_j - N_{j-1})$
(iii) $\rho N_{j-1} \geq N_j \geq 2N_{j-1}$
(iv) $N_j = n_j 2^{(1-\eta)j/\eta}$, with $2^j \leq n_j \leq \rho^j$.

We now construct N_{k+1} and M_{k+1} such that (i) to (iv) hold with $j = k+1$. For this, write $N_{k+1} = \rho_{k+1} N_k$, with $\rho_{k+1} \geq 2$ in order to satisfy the right inequality of condition (iii), and note that, in view of the inequality

$$M_{k+1} - M_k > c_4^{-1} N_{k+1}^{1/(1-\eta)} - c_4 N_k^{1/(1-\eta)},$$

deduced from (16) and (i), condition (ii) is satisfied as soon as $(c_4^{-1} \rho_{k+1}^{1/(1-\eta)} - c_4) N_k^{\eta/(1-\eta)} > 2^k(\rho_{k+1} - 1)$. As $n_k \geq 1$, the latter condition becomes $c_4^{-1} \rho_{k+1}^{1/(1-\eta)} - c_4 > \rho_{k+1} - 1$, which is satisfied for $\rho_{k+1} \geq \rho$. Hence, we can choose a real ρ_{k+1}, with $2 \leq \rho_{k+1} \leq \rho$, such that $N_{k+1} = \rho_{k+1} N_k$ and M_{k+1} determined by (i) satisfy conditions (i) to (iv).

Now, we define ψ_1 by putting

$$(17)\qquad \psi_1(r) = \frac{1}{4r^2} + \frac{1}{2^{k+1}(N_{k+1} - N_k)} \quad \text{if } N_k \le r < N_{k+1}.$$

We refer to [1] for the rest of the proof of the first part of the lemma; notice however that the factor $(4r^3)^{-1}$ occurring in inequalities (3.4) of [1] should be replaced by $(4rb_r^2)^{-1}$.

A precise definition of the sequence $(N_k)_k$ is crucial for the proof of the last statement of the lemma. Indeed, we are now able to control the jumps of the function ψ_1. Baker uses without proof the assumption that the function f is increasing on a neighborhood of $+\infty$, but this can fail to be true, especially when $N_k - N_{k-1}$ is very small compared with $N_{k+1} - N_k$. Hence our care.

Noticing that the function $x \mapsto x(4\log^{2\delta} x)^{-1}$ is increasing for x sufficiently large, we restrict our attention to the factor $2^{-k-1}(N_{k+1} - N_k)^{-1}$ occurring in (17). As it is constant on the intervals $[N_k, N_{k+1} - 1]$, we have to show that

$$(18)\qquad f(\exp\{N_{k+1}^{1/\delta}\}) > f(\exp\{N_k^{1/\delta}\})$$

for k sufficiently large, and to compare $f(\exp\{N_k^{1/\delta}\})$ with $f(\exp\{(N_k - 1)^{1/\delta}\})$. In order to prove (18), it is sufficient to note that condition (iii) implies that $2^{k+1}(N_{k+1} - N_k) < c_5 2^k(N_k - N_{k-1})$, for a constant $c_5 > 0$, independent of the integer k. Indeed, using $N_{k+1} \ge 2N_k$, we obtain

$$\frac{f(\exp\{N_{k+1}^{1/\delta}\})}{f(\exp\{N_k^{1/\delta}\})} \ge \frac{1}{c_5} \exp\{N_{k+1}^{1/\delta} - N_k^{1/\delta}\} \ge \frac{1}{c_5} \exp\{(2^{1/\delta} - 1)N_k^{1/\delta}\},$$

which tends to infinity with k.
The same argument shows that

$$f(\exp\{N_k^{1/\delta}\}) > (1/c_5) f(\exp\{(N_k - 1)^{1/\delta}\}),$$

and the proof of the last claim of the Lemma 3 is complete. □

Following Baker, we introduce the notion of $\sum$-sequence, first defined by J.W.S. Cassels [3].

Definition 1. Let $(a_r)_r$ be a strictly increasing sequence of positive integers and let μ_r be the number of fractions j/a_r $(0 < j < a_r)$ that are not equal to k/a_q $(q < r, k$ integer). Then $(a_r)_r$ is called a $\sum$-sequence if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \mu_r/a_r > 0.$$

For a proof of the next result, due to J.W.S. Cassels, see [3], Theorem IV.

Proposition 1. *If $(a_r)_r$ is a $\sum$-sequence and $\sum_{r=1}^{\infty} \psi(r)$ is a divergent series of decreasing positive terms, then the system of inequalities*

$$\{a_r x\} < \psi(r) \quad (r = 1, 2, \ldots)$$

has infinitely many solutions almost everywhere.

We deduce Theorem 1 from the following proposition.

Proposition 2. *Let $(a_r)_r$ be an increasing sequence of integers and $\delta \geq 1$ be a real number. Suppose that $(a_r)_r$ is a $\sum$-sequence and that there exist $c_6 > 0$ and $\eta > 0$ such that*

$$c_6^{-1} r^{(1-\eta)/\delta} < \log a_r < c_6 r^{(1-\eta)/\delta}.$$

Then $(a_r)_r$ is not a $L(\log^+ L)^{\delta - 1}$-sequence.

Proof. For $r \geq 1$, put $b_r = [2\log(4ra_r)] + 30$, then the sequence $(b_r^\delta)_r$ satisfies the hypothesis of Lemma 3 and there exist a divergent series $\sum_{r=1}^{\infty} \psi(r)$ and a function ψ_1 with the properties stated in Lemma 3.

Let g be the even function of period 1, defined by $g(0) = 0$ and

$$g(x) = \frac{1}{x}\psi_1\left(\log^\delta \frac{1}{x}\right), \quad \text{for } 0 < x \leq \frac{1}{2}.$$

Using the change of variables $z = \log^\delta(1/x)$, it follows from the convergence of $\sum_{r=1}^{\infty} \psi_1(r)$ that g belongs to $L(\log^+ L)^{\delta-1}$.
The function g being positive, we have for any integer j

$$R_{a_r}(g)(x) \geq \frac{1}{a_r} g\left(x + \frac{j}{a_r}\right).$$

As $(a_r)_r$ is a $\sum$-sequence and $\psi(r)/a_r$ decreases to 0 when r tends to $+\infty$, there are for almost all x infinitely many solutions of

$$0 < \left|x + \frac{j}{a_r}\right| < \frac{\psi(r)}{a_r} < \frac{1}{c_3} \quad (j \text{ integer}),$$

where c_3 is the constant occurring in Lemma 3.
Thus, applying the last part of Lemma 3 and (15), we have for almost every x

$$\begin{aligned} R_{a_r}(g)(x) &\geq \frac{1}{a_r}\frac{1}{c_2} g\left(\frac{\psi(r)}{a_r}\right) \\ &= c_2^{-1}\psi(r)^{-1}\psi_1((\log a_r - \log\psi(r))^\delta) \\ &\geq c_2^{-1}\psi(r)^{-1}\psi_1((\log a_r + \log(4rb_r^2))^\delta) \\ &\geq \frac{1}{c_2}\frac{\psi_1(b_r^\delta)}{\psi(r)}, \end{aligned}$$

for infinitely many r. Hence, using $\psi(r) = o(\psi_1(b_r))$, we get

$$\limsup_{r \to +\infty} R_{a_r}(f) = +\infty \quad \text{almost everywhere,}$$

and the proof is complete. $\square$

Proof of Theorem 1. Fix an integer $d \geq 2$ and a real number ε with $0 < \varepsilon < 1$. It is sufficient to build a sequence fulfilling the hypothesis of Proposition 2. To this end, denote by $p_1, \ldots, p_d$ distinct prime numbers and consider the sequence $(a_r)_r$ formed by all the numbers of the shape $p_1^{\alpha_1} \ldots p_d^{\alpha_d}$ in increasing order. First, we claim that $(a_r)_r$ is a $\sum$-sequence. Indeed, we have

$$\begin{aligned}\mu_r := \operatorname{Card}\left\{ j < a_r \;\middle|\; \frac{j}{a_r} \neq \frac{k}{a_q},\ q < r \text{ and } j \text{ integer} \right\} \\ \geq \operatorname{Card}\{ j < a_r \mid \text{none of the } p_i\text{'s divides } j \} \\ \geq \frac{a_r}{p_1 \ldots p_d},\end{aligned}$$

if r is sufficiently large.

Secondly, we have $\log a_r \sim c r^{1/d}$, for a constant $c > 0$ (see [9], Lemma 4.1). Hence, $(a_r)_r$ does not grow too rapidly and, applying Proposition 2 with the parameters $\delta = d - \varepsilon$ and $\eta = \varepsilon/d$, we infer that (a_r) is not a $L(\log^+ L)^{d-1-\varepsilon}$-sequence. Finally, it is easily seen that $(a_r)_r$ has dimension d and, consequently, is a $L(\log^+ L)^{d-1}$-sequence, which completes the proof of the corollary. □

4. SEQUENCES WITH INFINITE DIMENSION AND FINITE BREADTH

In the previous sections, we restricted our attention to sets of integers with finite dimension, hence, it remains to study sets with infinite dimension, like the set composed by all the prime numbers. Before reformulating Theorem A, stated in Section 1, and giving a proof of it, different from that of Rudin, we have to define the notion of *breadth*, first introduced by L. Dubins and J. Pitman [4].

Definition 2. We say that a set K of integers has breadth at most d if the least common multiple of every finite subset of K is the least common multiple of at most d elements of that subset. The least such d is called the breadth of K and, if no such d exists, we say that K has infinite breadth.

Theorem A. *Let (η_k) a strictly increasing sequence of integers with infinite breadth. Then there exist bounded measurable functions f on T such that $R_{n_k}(f)$ does not converge almost everywhere.*

Proof. We refer to [12] for an explicit construction of such a function f. By a different method and using a new deep result, J. Bourgain [2] recently rediscovered this result in the particular case when the sequence (n_k) contains all the prime numbers. It is not hard to show that his idea can be applied to obtain the whole result of Rudin.

Indeed, as (n_k) has infinite breadth, for every $r \geq 2$, there exist $k_1, \ldots, k_r$ such that n_{k_i} does not divide the least common multiple of $n_{k_1}, \ldots, n_{k_{i-1}}, n_{k_{i+1}}, \ldots, n_r$, for $1 \leq i \leq r$. Hence, there are $p_1, \ldots, p_r$ distinct prime numbers such that

$$v_{p_i}(n_{k_i}) > v_{p_i}(n_{k_j}), \qquad \text{whenever } i \neq j,$$

where $v_p(x)$ denotes the p-adic valuation of the integer x, i.e. the rational integer e such that p^e divides x but p^{e+1} does not.

Put $N = \mathrm{lcm}(n_{k_1}, \dots, n_{k_r})/(p_1 \dots p_r)$ and notice that n_{k_i} does not divide N for $1 \le i \le r$. Consider the set of integers

$$E = \{Np_1^{\alpha_1} \dots p_r^{\alpha_r} \mid \alpha_i \in \{0, 1\}\}$$

and the function

$$f = 2^{-r/2} \sum_{n \in E} e^{2i\pi nx}.$$

Then

$$R_{n_{k_s}}(f) = 2^{-r/2} \sum_{n \in (E \cap Np_s\mathbf{Z})} e^{2i\pi nx},$$

and, for $1 \le s \ne t \le r$,

$$\|R_{n_{k_s}}(f) - R_{n_{k_t}}(f)\|_2 = \frac{1}{\sqrt{2}}.$$

We conclude as in the proof of Proposition 4 of [2]. □

It is now natural to ask whether there exist sequences (n_k) with both infinite dimension and finite breadth, and then to study the convergence of $R_{n_k}(f)$. Such a sequence has been given explicitly by L. Dubins and J. Pitman ([4], Section 3b): denote by $p_1 < p_2 < \dots < p_k < \dots$ the sequence of primes and consider the set E_1 of all numbers of the type $p_1 \dots p_{j-1}\check{p}_j p_{j+1} \dots p_k$, for $k \ge 2$ and $1 \le j \le k$, where the symbol $\check{}$ means that p_j must be excluded.

Our aim is to exhibit, for fixed d, a sequence (n_k) with infinite dimension and finite breadth which is not a $L(\log^+ L)^d$-sequence. For this, in Lemma 5, we extend a theorem of R.C. Baker ([1], Theorem 3.2), which says that if (n_k) does not grow too rapidly, then (n_k) is not a L-sequence, and, in Lemma 4, we generalize the construction of L. Dubins and J. Pitman to obtain sequences with not too large growth.

Theorem 2. *For all $d \ge 0$, there exists a sequence with infinite dimension and breadth not exceeding $2d + 6$ which is not a $L(\log^+ L)^d$-sequence.*

The proof of the above theorem depends on the following two lemmas.

Lemma 4. *Let l be a positive integer. With the same notation as above, consider the set E_l of all integers n of the type*

$$n = p_1^{a_1} \dots p_{j-1}^{a_{j-1}} \check{p}_j p_{j+1} \dots p_k, \quad k \ge 2, \quad 1 \le j \le k$$
$$\text{and} \quad l \ge a_1 \ge \dots \ge a_{j-1} \ge 1.$$

Then E_l has infinite dimension and breadth not exceeding $l + 1$.

Proof. As the set E_l contains E_1, its dimension is obviously infinite. We consider a set $n_1, \ldots, n_r$ of r distinct elements of E_l and we shall extract from it a subset F with at most $l+1$ elements, such that the least common multiple of the elements of F is equal to the least common multiple of the n_i's. Let k be maximal such that p_k divides one of the n_i's and put in F a n_i for which the p_k-adic valuation is maximal, and, says, equal to j. Then let $k' < k$ be maximal such that there is a n_i satisfying $v_{p_{k'}}(n_i) > j$ and (if such k' exists!) put in F a n_i for which the $p_{k'}$-adic valuation is maximal. We repeat this process as often as we can. It stops after at most l steps, because no integer in F is divisible by the $(l+1)$-power of a prime. Hence, we have put at most l integers in the set F.

Finally, we note that there is at most one $k'' < k$ such that $v_{p_{k''}}(m) = 0$ for all m in F and $v_{p_{k''}}(n_1 \ldots n_r) \neq 0$ (observe that the construction implies that $v_{p_{k''}}(n_1 \ldots n_r) = 1$, precisely). Putting in F one of the n_i's with maximal $p_{k''}$-adic valuation, it is now clear that the set F satisfies the required property. □

Lemma 5. *If the sequence (n_k) satisfies the growth condition $n_k = O(\exp k^{1/(2d+5)})$, then (n_k) is not a $L(\log^+ L)^d$-sequence.*

Proof. We slightly extend the proof of Theorem 3.2 of R.C. Baker [1]. Applying a result of J.W.S. Cassels [3], also proved by Erdős and Koksma, we see that for almost all x the inequality

$$(19) \qquad 0 < \{n_k x\} \leq k^{-1/2} (\log k)^3$$

has infinitely many solutions k.
Let f be the even function of period 1, defined by $f(0) = 0$ and

$$f(x) = \frac{1}{x \log^{d+2}(1/x)}, \quad 0 < x \leq \frac{1}{2}.$$

Obviously, f belongs to $L(\log^+ L)^d$ and is decreasing on $]0, \exp(-d-2)]$.

Arguing as in Section 3 and using (19), for almost all x, there are infinitely many k such that

$$\begin{aligned} R_{n_k}(f)(x) &\geq n_k^{-1} f(n_k^{-1} k^{-1/2} (\log k)^3) \\ &\geq \frac{k^{1/2} (\log k)^{-3}}{\log^{d+2}(n_k k^{1/2} (\log k)^{-3})} \end{aligned}$$

and the latter function is greater than $k^{1/(5d+11)}$ if k is large enough. Thus, we get

$$\limsup_{k \to \infty} R_{n_k}(f)(x) = \infty$$

almost everywhere, and the lemma is proved. □

Proof of Theorem 2. We want to apply Lemma 5 to the sequences E_l, so we need a precise estimate of their growth. Let $l \geq 1$ and $s \geq 2$ be integers. In order to count the number, say $\psi(s, l)$, of elements of E_l less than $p_1^l \ldots p_{s+1}^l$, denote by $\pi(s, l)$ the cardinality of the set of integers

$$n = p_1^{a_1} \dots p_s^{a_s}, \quad 1 \le a_s \le \dots \le a_1 \le l.$$

Distinguishing between the l values that can take a_1, we obtain the recursion formula

$$\pi(s, l) = \pi(s-1, l) + \dots + \pi(s-1, 1),$$

and an easy induction gives

$$\pi(s, l) \ge \frac{s^{l-1}}{(l-1)!}.$$

Further, the cardinality of the set of integers

$$n = p_1^{a_1} \dots p_{j-1}^{a_{j-1}} \check{p}_j p_{j+1} \dots p_{s+1}, \quad 1 \le a_{j-1} \le \dots \le a_1 \le l$$

equals to $\pi(s, l) + \dots + \pi(1, l)$, which is not less than $s^l/l!$. From the definition of the sequence $E_l := (n(k))_{k \ge 1}$, we deduce that

$$\psi(s, l) \ge \frac{1}{l!} + \frac{2^l}{l!} + \dots + \frac{s^l}{l!} \ge \frac{s^{l+1}}{(l+1)!}$$

whence, applying Theorem 8 and Theorem 415 of [5], there is a $c > 0$ such that

$$n(s^{l+1}/(l+1)!) \le p_1^l \dots p_{s+1}^l \le \exp\{cls \log s\}.$$

Finally, there exists a constant $c(l)$, depending only on l, such that

$$n(k) \le \exp\{c(l) k^{1/(l+1)} \log k\}, \quad k \ge 1.$$

We can now complete the proof. Let $d \ge 0$ be an integer and put $l = 2d + 5$. Then, applying Lemma 4 and Lemma 5, we claim that the set E_l has infinite dimension and finite breadth, but is not a $L(\log^+ L)^d$-sequence. $\square$

We conclude this section by proving some positive results for the sequence E_1.

Proposition 3. *Let $f = \sum_{\nu=0}^{\infty} a_\nu e_\nu$, where $\{a_\nu, \nu \ge 0\} \in l_2$ satisfies*

(20) $$\sum_{\nu=0}^{\infty} a_\nu^2 \left(\frac{\log l}{\log \log l} \right) < \infty.$$

Then,

(21) $$m\left\{ \lim_{E_1 \ni n \to \infty} R_n(f) = \int_{\mathbb{T}} f \, dm \right\} = 1,$$

where the set of numbers E_1 is defined in Lemma 4. As concerning averaging along E_1, writing $E_1 = \{n_k, k \ge 1\}$

(22) $$m\left\{ \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} R_{n_k}(f) = \int_{\mathbb{T}} f \, dm \right\} = 1,$$

holds for all $f \in L^2(m)$.

Proof. Let $t > 0$ and k_o be fixed. Then

$$m\left\{\sup_{\substack{1\le j\le k+1\\ k\ge k_o}} |R_{p_1\cdots\check{p}_j\cdots p_{k+1}}(f) - R_{p_1\cdots p_{k+1}}(f)| > t\right\}$$

$$\le \frac{1}{t^2}\sum_{\substack{1\le j\le k+1\\ k\ge k_o}} \int \left|\sum_{\substack{p_1\cdots\check{p}_j\cdots p_{k+1}|l\\ (p_j,l)=1}} a_l e_l\right|^2 dm$$

$$\le \frac{1}{t^2}\sum_{\substack{1\le j\le k+1\\ k\ge k_o}} \sum_{\substack{p_1\cdots\check{p}_j\cdots p_{k+1}|l\\ (p_j,l)=1}} a_l^2.$$

Given an arbitrary number l, if $k_2 > k_1 \ge k_o$ are such that

$$p_1\cdots\check{p}_{j_1}\cdots p_{k_1+1}\,|\,l,\quad p_{j_1}\not| \, l,\quad p_1\cdots\check{p}_{j_2}\cdots p_{k_2+1}\,|\,l,\quad p_{j_2}\not| \, l,$$

then $j_1 = j_2$. Defining thus $k(l)$ as being the index corresponding to the smallest j such that p_j does not divide l, we get

$$m\left\{\sup_{\substack{1\le j\le k+1\\ k\ge k_o}} |R_{p_1\cdots\check{p}_j\cdots p_{k+1}}(f) - R_{p_1\cdots p_{k+1}}(f)| > t\right\}$$

$$\le \frac{1}{t^2}\sum_{l\ge p_1\cdots p_{k_o}} (k(l) - k_o - 1)a_l^2.$$

But $l \ge p_1\cdots p_{k(l)-2}$, which gives

$$k(l) = O\left(\frac{\log l}{\log\log l}\right),$$

and allows us to conclude the first half of the proposition. Concerning the second half, observe that

$$\int\left(\frac{1}{N^2}\sum_{\substack{j\le k+1\\ k\le N}} [R_{p_1\cdots\check{p}_j\cdots p_{k+1}}(f) - R_{p_1\cdots p_{k+1}}(f)]\right)^2 dm$$

$$= \frac{1}{N^4}\sum_{l=0}^{\infty} a_l^2\,\mathrm{Card}\{j\le k+1, k\le N\,|\,p_j\not| \, l, p_1\cdots\check{p}_j\cdots p_{k+1}|\,l\}^2$$

$$\le \frac{(N-k(l))^2}{N^4} \le \frac{1}{N^2}.$$

Therefore,

$$\sum_{N\ge 1}\int\left(\frac{1}{N^2}\sum_{\substack{j\le k+1\\ k\le N}} [R_{p_1\cdots\check{p}_j\cdots p_{k+1}}(f) - R_{p_1\cdots p_{k+1}}(f)]\right)^2 dm < \infty,$$

which, combined with Jessen's theorem implies

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{\substack{j \leq k+1 \\ k \leq N}} R_{p_1 \cdots \check{p}_j \cdots p_{k+1}}(f) = \int f \, dm,$$

and this easily allows us to get the second half of the proposition. □

5. OPEN QUESTIONS

Since we have very few information about sequences of infinite dimension, the following questions arise naturally.

Does there exist some sequence with infinite dimension, which is an L^p-sequence for some (or all) $p > 1$?

Suppose the sequence $\mathcal{S}$ has finite breadth. Is it true that $\mathcal{S}$ is an L^p-sequence for some $1 \leq p \leq \infty$? Does there exist an explicit connexion between breadth and p?

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